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SINGULAR PERTURBATIONS USED TO SOLVE THE EQUATIONS OF THERMOELASTICITY IN A CROSS-SECTION OF A ROTATING CYLINDER†

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An analysis is made of the two-dimensional stressed state in a cross-section of a cylinder rotating at constant angular velocity and experiencing thermal disturbances due to convective and radiative heat exchange with the external medium. The stress equations are used in the approximation of the uncoupled theory of thermoelasticity. Asymptotic formulae are derived for the stresses in terms of the small parameter $1/\sqrt{(\mathrm{Pd})}$ (where Pd is the Predvoditelev criterion). They enable one to take into account the varying nature of the heat-exchange coefficients along the perimeter of the cylinder, to form a qualitative picture of the stress distribution in the boundary-layer strip and in the rest of the domain, and to show that the stress state is determined by the non-axisymmetrical part of the temperature field.

IN A PREVIOUS study [1]‡ one of us constructed formulae to calculate the temperature in a cross-section of a cylinder of radius R, rotating at a fixed angular velocity ω and interacting with the external medium as governed by the law of radiative and convective heat exchange. We shall derive asymptotic formulae for the thermal stresses that arise in the cylinder, using the uncoupled theory of thermoelasticity [2].

In the laboratory system of polar coordinates (ρ, φ) under quasi-steady conditions, the temperature $T(\rho, \varphi)$ at a point of the cross-section with physical coordinates $(R\rho, \varphi)$ satisfies the equation [1]

$$\partial T/\partial\varphi=\varepsilon^2\Delta_{\rho,\varphi}T,\quad 0<\rho<1,\quad \varphi\in[0,2\pi] \tag{0.1}$$

$$\Delta_{\rho,\varphi}=\rho^{-1}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\rho^{-2}\frac{\partial^2}{\partial\varphi^2}$$

the boundary condition

$$\frac{\partial}{\partial\rho}T+b(\varphi)T+\sigma(\varphi)\beta(T)=f(\varphi),\quad \rho=1,\quad \varphi\in[0,2\pi] \tag{0.2}$$

and the condition of continuity as $\rho\to 0$

$$\lim_{\rho\to 0}\rho\frac{\partial T}{\partial\rho}=0,\quad \varphi\in[0,2\pi]$$

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‡LETAVIN M. I., On a singular problem of temperature control for a rotating cylinder in a quasi-steady regime. Vologda, 1989. Unpublished paper, deposited at VINITI 9.02.89, No. 895-B89.

where

$$\varepsilon^2 = (\mathrm{Pd})^{-1} = a/(\omega R^2), \quad b(\varphi) = \alpha_k(\varphi)R/\lambda$$
$$\sigma(\varphi) = \sigma_r(\varphi)R/\lambda, \quad \beta(T) = T^4$$
$$f(\varphi) = b(\varphi)T_k(\varphi) + \sigma(\varphi)\beta(T_r(\varphi)) \tag{0.3}$$

where a is the thermal diffusivity, Pd is the Predvoditelev criterion, $b(\varphi)$ is the Biot criterion, λ is the thermal conductivity, $\alpha_k(\varphi)$ is the heat transfer coefficient, $T_k(\varphi)$ is the temperature of the medium with which the cylinder is exchanging heat convectively, $\sigma_r(\varphi)$ is the radiative heat exchange coefficient and $T_r(\varphi)$ is the temperature of the medium with which the cylinder is exchanging heat by radiation.

The stresses in the cylinder cross-section will be determined in the two-dimensional deformed state approximation [2], using the stress function $F(\rho, \varphi)$, by the formulae

$$\sigma_\rho = \rho^{-1}\frac{\partial}{\partial\rho}F + \rho^{-2}\frac{\partial^2}{\partial\varphi^2}F, \quad \sigma_\varphi = \frac{\partial^2}{\partial\rho^2}F, \quad \sigma_{\rho\varphi} = -\frac{\partial}{\partial\rho}\rho^{-1}\frac{\partial}{\partial\varphi}F \tag{0.4}$$

while F itself will be determined as the solution of the equation of the uncoupled quasi-steady theory of thermoelasticity

$$\Delta^2_{\rho,\varphi}F + C\Delta_{\rho,\varphi}T = 0, \quad 0 < \rho < 1, \quad \varphi \in [0, 2\pi] \tag{0.5}$$

satisfying the boundary conditions on the free surface

$$F = \partial F/\partial\rho = 0, \quad \rho = 1, \quad \varphi \in [0, 2\pi) \tag{0.6}$$

and the boundedness conditions for the derivatives appearing in (0.4).

In Eq. (0.5) $C = \alpha_T(1-\nu)^{-1}E$, α_T is the coefficient of linear thermal expansion, ν is Poisson's ratio, and E is Young's modulus.

The coefficient ε in Eq. (0.1) is a small parameter in the case of cylinders and rollers in metallurgical machinery [3]. this fact will be used to construct a solution of system (0.1), (0.2), (0.5) and (0.6).

From now on all functions considered will be 2π-periodic in φ.

1. CONSTRUCTION OF A FORMAL ASYMPTOTIC EXPANSION

According to the general theory [4], we are looking for an expansion of the function T in the form

$$T(\varepsilon,\rho,\varphi) = U(\varepsilon,\rho,\varphi) + V(\varepsilon,r,\varphi) = \sum_{n=0}^{\infty}\varepsilon^n u_n(\rho,\varphi) + \sum_{n=1}^{\infty}\varepsilon^n \upsilon_n(r,\varphi), \quad r = \frac{1-\rho}{\varepsilon} \tag{1.1}$$

where U and V are the regular and boundary-layer parts of the expansion, so that $|\upsilon_n(r,\varphi)| < C_1\exp(-C_2 r)$, $r > 0$, C_1, $C_2 > 0$.

We know [1] that the functions $u_n = \mathrm{const}$ and υ_n are found from the equations

$$\langle b(\varphi)\rangle u_0 + \langle\sigma(\varphi)\rangle\beta(u_0) = \langle f(\varphi)\rangle \tag{1.2}$$

$$\langle b(\varphi) + \sigma(\varphi)\beta'(u_0)\rangle u_n = -\langle \upsilon_n(0,\varphi)(b(\varphi) + \sigma(\varphi)\beta'(u_0))\rangle + \langle P_n(\varphi)\rangle, \quad n = 1, 2, \ldots \tag{1.3}$$

$$M\upsilon_n \equiv \left(\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial r^2}\right)\upsilon_n = -\sum_{k=1}^{n-1} r^{n-1-k}\frac{\partial}{\partial r}\upsilon_k +$$

$$+\sum_{k=1}^{n-2}(n-2-k)r^{n-2-k}\frac{\partial^2}{\partial\varphi^2}\upsilon_k,\quad r>0,\quad \varphi\in[0,2\pi) \tag{1.4}$$

$$\frac{\partial \upsilon_1}{\partial r}=b(\varphi)u_0+\sigma(\varphi)\beta(u_0)-f(\varphi),\quad r=0,\quad \varphi\in[0,2\pi) \tag{1.5}$$

$$\frac{\partial \upsilon_n}{\partial r}=(b(\varphi)+\sigma(\varphi)\beta'(u_0))(u_{n-1}+\upsilon_{n-1})+P_{n-1}(\varphi),\quad r=0,\quad \varphi\in[0,2\pi)$$

$$P_n=\sum_{k=2}^{n}\beta^{(k)}(u_0)q_{n,k},\quad \langle P_n(\varphi)\rangle=\frac{1}{2\pi}\int_0^{2\pi}P_n(\varphi)d\varphi$$

where $q_{n,k}$ are polynomials in the variables $u_l+\upsilon_l(0,\varphi)$ $(l=1,\ldots,n-1)$ of degree k. The sum in (1.4)—and throughout what follows—is, by definition, zero if the lower index of summation exceeds the upper one.

Under the conditions

$$b(\varphi),\quad \sigma(\varphi)\geqslant 0,\quad \langle b(\varphi)\rangle+\langle\sigma(\varphi)\rangle>0 \tag{1.6}$$

$$T_r(\varphi),\quad T_k(\varphi)>0 \tag{1.7}$$

Eq. (1.2) has a unique positive solution $u_0>0$; It then follows from (1.6) that $\langle b(\varphi)+\sigma(\varphi)\beta(u_0)\rangle>0$ and u_n $(n=1,2,\ldots)$ are then uniquely determined from Eqs (1.3) Thus, the terms of the asymptotic series (1.1) are found successively: u_0 from (1.2), then υ_1 from (1.4) and (1.5), u_1 from (1.3), υ_2 from (1.4) and (1.5); and so on.

By analogy with (1.1), we construct an expansion of $F(\rho,\varphi)$ as

$$F(\varepsilon,\rho,\varphi)=H(\varepsilon,\rho,\varphi)+G(\varepsilon,r,\varphi)=\sum_{n=2}^{\infty}\varepsilon^n h_n(\rho,\varphi)+\sum_{n=3}^{\infty}\varepsilon^n g_n(r,\varphi) \tag{1.8}$$

$$|g_n(r,\varphi)|<C_1\exp(-C_2 r),\quad r>0 \tag{1.9}$$

where H is the regular part and G the boundary-layer part of the expansion.

Assuming that Eq. (0.5) holds separately for the regular part of the expansion H, we obtain equations for h_n

$$\Delta^2_{\rho,\varphi}h_n=0,\quad 0<\rho<1,\quad \varphi\in[0,2\pi),\quad n=2,3,\ldots \tag{1.10}$$

For the boundary-layer part G, we rewrite Eq. (0.5) in the form $\Delta_{\rho,\varphi}(\Delta_{\rho,\varphi}G+CV)=0$ and satisfy the simpler equation $\Delta_{\rho,\varphi}G+CV=0$, from which we obtain equations for g_n

$$\frac{\partial^2}{\partial r^2}g_n=-C\upsilon_{n-2}-Q_n,\quad n=3,4,\ldots,\quad r>0,\quad \varphi\in[0,2\pi) \tag{1.11}$$

$$Q_n=-\sum_{k=4}^{n}r^{n-k}\frac{\partial}{\partial r}g_{k-1}+\sum_{k=5}^{n}(n-k)r^{n-k}\frac{\partial^2}{\partial\varphi^2}g_{k-2} \tag{1.12}$$

We will write the solution of Eq. (1.11) satisfying an estimate of the form (1.9) as follows:

$$g_n(r,\varphi)=-\int_r^{\infty}(\varsigma-r)[C\upsilon_{n-2}(\varsigma,\varphi)+Q_n(\varsigma,\varphi)]d\varsigma,\quad n=3,4,\ldots,r>0,\quad \varphi\in[0,2\pi] \tag{1.13}$$

Substituting the expansion (1.8) into the boundary condition (0.6), we obtain the boundary conditions for the functions h_n

$$h_2(1,\varphi)=0,\quad h_n(1,\varphi)=-g_n(0,\varphi),\quad n=3,4\ldots,\quad \varphi\in[0,2\pi) \tag{1.14}$$

$$\frac{\partial}{\partial\rho}h_n(1,\varphi)=-\frac{\partial}{\partial r}g_{n+1}(0,\varphi),\quad n=2,3,\ldots,\quad \varphi\in[0,2\pi) \tag{1.15}$$

One can then successively determine the coefficients of (1.8): g_3 from (1.13), then h_2 as a bounded solution of problem (1.10), (1.14) and (1.15), g_4 from (1.13), then h_3 from (1.10), (1.14) and (1.15), and so on.

It follows from the way in which the parts of the expansion (1.8) were determined that G is the thermoelastic part and H the elastic part of the series for the stress function [2].

2. JUSTIFICATION OF THE ASYMPTOTIC EXPANSION

In Cartesian coordinates $x=(x_1, x_2)$, $x_1=\rho\cos\varphi$, $x_2=\rho\sin\varphi$, we put $\Omega=\{(x_1, x_2): x_1^2+x_2^2<1\}$, $\Gamma=\overline{\Omega}/\Omega$. The following asymptotic approximations will be taken for T and F

$$T_N(\varepsilon,x)=T_N(\varepsilon,\rho\cos\varphi,\rho\sin\varphi)=U_N(\varepsilon)+\Psi(\rho)V_N(\varepsilon,r,\varphi)=$$
$$=\sum_{n=0}^{N}\varepsilon^n u_n+\Psi(\rho)\sum_{n=1}^{N}\varepsilon^n \upsilon_n(r,\varphi) \tag{2.1}$$

$$F_N(\varepsilon,x)=F_N(\varepsilon,\rho\cos\varphi,\rho\sin\varphi)=H_N(\varepsilon,\rho,\varphi)+\Psi(\rho)G_N(\varepsilon,r,\varphi)=$$
$$=\sum_{n=2}^{N+1}\varepsilon^n h_n(\rho,\varphi)+\Psi(\rho)\sum_{n=3}^{N+2}\varepsilon^n g_n(r,\varphi),\quad N=0,1,2,\dots \tag{2.2}$$

where $\Psi(\rho)\in C^\infty([0, 1)]$, $\Psi(\rho)=0$ and $\rho\in[0, 1/3]$, $\Psi(\rho)=1$ for $\rho\in[2/3, 1]$, $0\leqslant\Psi(\rho)\leqslant 1$. The residuals of the exact solutions $T(\varepsilon, x)$ and $F(\varepsilon, x)$ and their approximations (2.1) and (2.2)

$$\delta T_N(\varepsilon,x)=T(\varepsilon,x)-T_N(\varepsilon,x),\quad \delta F_N(\varepsilon,x)=F(\varepsilon,x)-F_N(\varepsilon,x) \tag{2.3}$$

are solutions of the equations

$$-\varepsilon^2\Delta_x(\delta T_N)+b_i(\delta T_N)_{x_i}=-W_1,\quad x\in\Omega \tag{2.4}$$

$$b_1(x)=-x_2,\quad b_2(x)=x_1,\quad \Delta_x=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$$

$$\Delta_x^2(\delta F_N)+C\Delta_x(\delta T_N)=-W_2,\quad x\in\Omega \tag{2.5}$$

satisfying the boundary conditions

$$\left(\frac{\partial}{\partial\nu}+\alpha(x)+\sigma(x)\beta'(u_0)\right)\delta T_N=-W_3,\quad x\in\Gamma \tag{2.6}$$

$$\delta F_N=-\varepsilon^{N+2}g_{N+2},\quad \frac{\partial}{\partial\nu}(\delta F_N)=0,\quad x\in\Gamma \tag{2.7}$$

where

$$W_1(\varepsilon,x)=(\partial/\partial\varphi-\varepsilon^2\Delta_{\rho,\varphi})(\psi(\rho)V_N(\varepsilon,r,\varphi))$$
$$W_2(\varepsilon,x)=\Delta_x^2(\psi(\rho)G_N(\varepsilon,r,\varphi))+C\Delta_x(\psi(\rho)V_N(\varepsilon,r,\varphi))$$
$$W_3(\varepsilon,x)=W_3(\varepsilon,\cos\varphi,\sin\varphi)=\frac{\partial}{\partial\rho}T_N(\varepsilon,1,\varphi)+$$
$$+\alpha(\varphi)T_N(\varepsilon,1,\varphi)+\sigma(\varphi)\beta(u_0)-f(\varphi)+\sigma(\varphi)\{\beta(T_N(\varepsilon,1,\varphi)+$$
$$+\omega^*)-\beta'(u_0)\omega^*-\beta(u_0)\},\quad \omega^*=\delta T_N$$

$\nu=\nu(x)$ is the unit vector along the outward normal to Γ at x.

We introduce the auxiliary function by

$$\overset{*}{g}_{N+2}(x)=\overset{*}{g}_{N+2}(\rho\cos\varphi,\rho\sin\varphi)=g_{N+2}(0,\varphi,)\Psi(\rho),\quad x\in\Omega \tag{2.8}$$

and set

$$y(x)=\delta F_N+\varepsilon^{N-2}\overset{*}{g}_{N+2} \tag{2.9}$$

We then obtain an equation for $y(x)$ from (2.5)

$$\Delta_x^2 y=-C\Delta_x(\delta T_N)-W_2+\varepsilon^{N+2}\Delta_x^2\overset{*}{g}_{N+2},\quad x\in\Omega \tag{2.10}$$

with zero boundary conditions

$$y=0,\quad \partial y/\partial\nu=0,\quad x\in\Gamma \tag{2.11}$$

We now estimate $\|y\|_{W_2^2(\Omega)}$. Multiplying Eq. (2.10) by y, integrating by parts over Ω subject to condition (2.11) and using the form of the function W_2, we write

$$\int_\Omega(\Delta_x y)^2dx=\varepsilon^{N+2}\int_\Omega(\Delta_x y)(\Delta_x\overset{*}{g}_{N+2})dx-C\int_\Omega(\Delta_x y)\delta T_N dx-$$

$$-\int_\Omega(\Delta_x y)(\Delta_x(\Psi G_N)+C\Psi V_N)dx \tag{2.12}$$

Using Cauchy's inequality and the inequality [5]

$$\|y\|_{W_2^2(\Omega)}\leqslant A_1\left(\int_\Omega(\Delta_x y)^2dx\right)^{1/2}$$

which holds by conditions (2.11), we derive the following estimate from (2.12)

$$\|y\|_{W_2^2(\Omega)}\leqslant A_1(\varepsilon^{N+2}\|\Delta_x\overset{*}{g}_{N+2}\|_{L^2(\Omega)}+C\|\delta T_N\|_{L^2(\Omega)}+$$

$$+\|\Delta_x(\Psi G_N)+C\Psi V_N\|_{L^2(\Omega)})=A_1(\varepsilon^{N+2}I_1+CI_2+I_3) \tag{2.13}$$

We must now estimate I_1, I_2 and I_3 in (2.13) To estimate I_3, we use the definitions of G_N and V_N in (2.1) and (2.2) and write

$$\Delta_x(\psi G_N)+C\Psi V_N=\Psi'(\rho)(\rho^{-1}G_N(\varepsilon,r,\varphi)+2\frac{\partial}{\partial\rho}G_N(\varepsilon,r,\varphi)+$$

$$+\Psi''(\rho)G_N(\varepsilon,r,\varphi)+\varepsilon^{N+1}\Psi(\rho)\Bigg[-\rho^{-1}\sum_{n=2}^{N+1}r^{N-n+1}\frac{\partial}{\partial r}g_{n+1}+$$

$$+\rho^{-2}\sum_{n=3}^{N+3}r^{N-n+1}\left(\frac{\partial^2}{\partial\varphi^2}g_n\right)(1+(N-n+1)\rho)\Bigg] \tag{2.14}$$

Using the properties of $\Psi(\rho)$, we deduce from (2.14) that

$$I_3\leqslant\varepsilon^{N+1}A_2\sum_{n=1}^{N+1}\left\|\Psi(\rho)r^{N-n}\left(r\frac{\partial}{\partial r}g_{n+1}+\frac{\partial^2}{\partial\varphi^2}g_{n+1}\right)\right\|_{L^2(\Omega)}+$$

$$+A_3\sum_{n=3}^{N+2}\left\{\|\Psi'(\rho)g_n(r,\varphi)\|_{L^2(\Omega)}+\left\|\Psi'(\rho)\frac{\partial}{\partial r}g_n(r,\varphi)\right\|_{L^2(\Omega)}+\|\Psi''(\rho)g_n(r,\varphi)\|_{L^2(\Omega)}\right\} \tag{2.15}$$

Assuming that the functions g_n satisfy the estimates

$$\int_0^{2\pi} g_n^2(r,\varphi)+\left(\frac{\partial}{\partial r}g_n(r,\varphi)\right)^2+\left(\frac{\partial^2}{\partial\varphi^2}g_n(r,\varphi)\right)^2 d\varphi\leq A_4\exp(-A_5 r) \tag{2.16}$$

($r\geq 0$, $n=3,\ldots,N+2$), we infer from (2.15) that

$$I_3\leq\varepsilon^{N+1}A_6+A_7\exp\left(-A_5(3\varepsilon)^{-1}\right)\leq\varepsilon^{N+1}A_8 \tag{2.17}$$

where A_6, A_7 and A_8 depend only on A_4, A_5 and N.

To estimate I_1, we note that, by (2.8)

$$\Delta_x g^*_{N+2}=\Delta_{\rho,\varphi}\left(g_{N+2}(0,\varphi)\Psi(\rho)\right)=g_{N+2}(0,\varphi)\rho^{-1}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\Psi(\rho)\right)+$$
$$+\rho^{-2}\Psi(\rho)\frac{\partial^2}{\partial\varphi^2}g_{N+2}(0,\varphi) \tag{2.18}$$

Therefore

$$I_1\leq A_9 \tag{2.19}$$

where A_9 depends only on A_4 (see (2.16))

To estimate I_2 and justify inequality (2.16), we need a corollary of Sec. 5 in the paper cited in the footnote.

Lemma 1. Let the functions $b(\varphi)$, $\sigma(\varphi)$, $T_k(\varphi)$, $T_r(\varphi)$ in (0.2), (0.3) be 2π-periodic and absolutely continuous; assume that their derivatives are of bounded variation in $[0, 2\pi]$ and that conditions (1.6) and (1.7) are satisfied. Then for any integer $N_1\geq 0$ there exists $\varepsilon_0>0$ such that, for $\varepsilon\in(0,\varepsilon_0]$ ($N=0,1,\ldots,N_1$), the residual δT_N defined in (2.3) and satisfying Eqs (2.4) and (2.6) also satisfies the estimate

$$\|\delta T_N\|_{L^2(\Omega)}\leq A_{10}\varepsilon^{N+1} \tag{2.20}$$

and the functions $v_n(r,\varphi)$ ($n=1,2,\ldots,N_1$) defined by Eqs (1.4) and (1.5) satisfy the inequalities

$$\int_0^{2\pi} v_n^2(r,\varphi)+\left(\frac{\partial}{\partial r}v_n(r,\varphi)\right)^2+\left(\frac{\partial^2}{\partial\varphi^2}g_n(r,\varphi)\right)^2 d\varphi\leq A_{11}\exp(-A_{12}r),\ r\geq 0 \tag{2.21}$$

Estimate (2.16) follows from (2.21) in Lemma 1 and formulae (1.12) and (1.13) (the definition of $g_n(r,\varphi)$). Inequalities (2.17), (2.19) and (2.20), applied to (2.13), give

$$\|y\|_{W_2^2(\Omega)}\leq\varepsilon^{N+1}A_{13},\ \varepsilon\in(0,\varepsilon_0] \tag{2.22}$$

Using Eq (2.9) and inequality (2.22) we obtain

$$\|\delta F_N\|_{W_2^2(\Omega)} \leqslant \|y\|_{W_2^2(\Omega)} + \varepsilon^{N+2}\|\overset{*}{g}_{N+2}\|_{W_2^2(\Omega)} \leqslant$$
$$\leqslant A_{13}\varepsilon^{N+1} + A_{14}\varepsilon^{N+2} \leqslant A_{15}\varepsilon^{N+1}, \quad \varepsilon \in (0, \varepsilon_0] \tag{2.23}$$

where the bound

$$\|\overset{*}{g}_{N+2}\|_{W_2^2(\Omega)} \leqslant A_{14}$$

is established by using a representation of type (2.18) for the derivatives of $\overset{*}{g}_{N+2}$.

We can thus formulate the following estimation theorem.

Theorem 1. With the same assumptions as in Lemma 1, for any integer $N_1 \geqslant 0$ there exists $\varepsilon_0 > 0$ such that the stress function $F(\varepsilon, x)$ admits of an asymptotic representation (2.3), where $\delta F_N(\varepsilon, x)$ satisfies inequality (2.23) with a constant A_{15} independent of $N = 0, 1, \ldots, N_1$, $\varepsilon \in (0, \varepsilon_0]$.

3. ASYMPTOTIC FORMULAE FOR THE STRESSES

Using (0.4), we can write the principal terms of the asymptotic expansions for the stresses

$$\sigma_\rho(\rho,\varphi) = \varepsilon^2\left[\rho^{-1}\left(\frac{\partial}{\partial\rho} + \rho^{-1}\frac{\partial^2}{\partial\varphi^2}\right)h_2(\rho,\varphi) - \frac{\partial}{\partial r}g_3(r,\varphi)\right] \tag{3.1}$$

$$\sigma_\varphi(\rho,\varphi) = \varepsilon\frac{\partial^2}{\partial r^2}g_3(r,\varphi) + \varepsilon^2\frac{\partial^2}{\partial\rho^2}h_2(\rho,\varphi) \tag{3.2}$$

$$\sigma_{\rho\varphi}(\rho,\varphi) = \varepsilon^2\left[-\frac{\partial}{\partial\rho}\left(\rho^{-1}\frac{\partial}{\partial\varphi}h_2(\rho,\varphi)\right) + \frac{\partial^2}{\partial r\partial\varphi}g_3(r,\varphi)\right] \tag{3.3}$$

where $r = (1-\rho)/\varepsilon$, formulae (3.1) and (3.3) are of accuracy $O(\varepsilon^3)$, and formula (3.2) is of accuracy $O(\varepsilon^2)$ in a boundary layer of thickness of $O(\varepsilon^2)$, and of accuracy $O(\varepsilon^3)$ in the remainder of Ω. The solution of Eqs (1.4) and (1.5) for υ_1 may be expressed as follows [6]:

$$\upsilon_1(r,\varphi) = \sum_{k=1}^{\infty}\upsilon_k^+(r)\cos k\varphi + \upsilon_k^-(r)\sin k\varphi$$

$$\upsilon_k^\pm(r) = \exp\left(-\sqrt{k/2}r\right)\left\{\pm\left(\varsigma_k^s \mp \varsigma_k^c\right)\cos\left(\sqrt{k/2}r\right) + \left(\varsigma_k^s \pm \varsigma_k^c\right)\sin\left(\sqrt{k/2}r\right)\right\}/\sqrt{2k}$$

where ζ_k^c, ζ_k^s are the Fourier coefficients of $\zeta(\varphi) = b(\varphi)u_0 + \sigma(\varphi)\beta(u_0) - f(\varphi)$. Expressing the function g_3 in accordance with formula (1.13) and determining h_2 from Eqs (1.10), (1.14) and (1.15) we can write the stresses (3.1)–(3.3) as follows:

$$\sigma_\rho(\rho,\varphi) = \varepsilon^2 C\left\{\rho\varsigma_1^c(\cos\varphi - \sin\varphi) + \frac{1}{2}\sum_{k=2}^{\infty}k^{-1}\varsigma_k^c\rho^{k-2} \times\right.$$
$$\times\left(k(k-1)\left(1-\rho^2\right) + 2\rho^2\right)(\cos k\varphi - \sin k\varphi) -$$
$$\left.-\sum_{k=1}^{\infty}g_k^+(r)\cos k\varphi + g_k^-(r)\sin k\varphi\right\} \tag{3.4}$$

$$\sigma_\varphi(\rho,\varphi) = -\varepsilon C\upsilon_1(r,\varphi) + \varepsilon^2 2^{-1}C\left\{6\rho\varsigma_1^c(\cos\varphi - \sin\varphi) +\right.$$

$$+\sum_{k=2}^{\infty}\varsigma_k^c k^{-1}\rho^{k-2}((k+2)(k+1)\rho^2 - k(k-1))(\cos k\varphi - \sin k\varphi)\Big\}$$

$$\sigma_{\rho\varphi}(\rho,\varphi)=\varepsilon^2 C\Big\{\rho\varsigma_1^c(\cos\varphi+\sin\varphi)+2^{-1}\sum_{k=2}^{\infty}\varsigma_k^c\rho^{k-2}((k+1)\rho^2-k+1)\times$$

$$\times(\cos k\varphi+\sin k\varphi)+\sum_{k=1}^{\infty}k(g_k^-(r)\cos k\varphi - g_k^+(r)\sin k\varphi)\Big\} \qquad (3.5)$$

$$g_k^{\pm}(r)=k^{-1}\exp(-\sqrt{k/2r})\Big\{\pm\varsigma_k^c\cos\big(\sqrt{k/2r}\big)+\varsigma_k^s\sin\big(\sqrt{k/2r}\big)\Big\} \qquad (3.6)$$

By Saint-Venant's principle, if the front surface of the cylinder is free of loads, we obtain the following expression for the axial stresses $\sigma_z(\rho, \varphi)$ in the cylinder cross-section, accurate to within $O(\varepsilon^2)$.

$$\sigma_z = -\varepsilon C \upsilon_1(r,\varphi) \qquad (3.7)$$

4. CALCULATION OF THERMAL STRESSES IN A ROLLER OF A CONTINUOUS BLANK CASTING

The rollers of a continuous blank casting, being in contact with incandescent metal, from which they receive a powerful heat flux due to both direct contact and radiation, experience substantial thermal stresses.

Let us apply the above computational formulae to calculate the thermoelastic stresses in a solid roller of diameter 0.38 m, for a machine operating in the converter department of the Cherepovets Metallurgical Complex, where the necessary natural measurements were carried out.

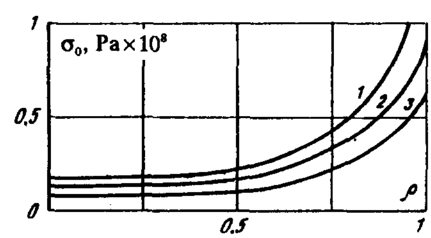

FIG.1.

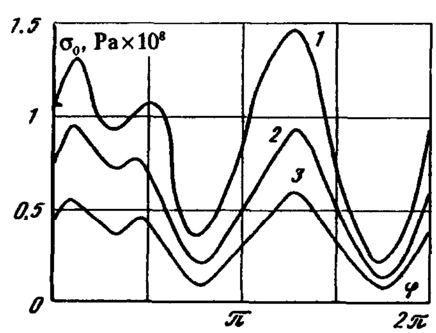

FIG.2.

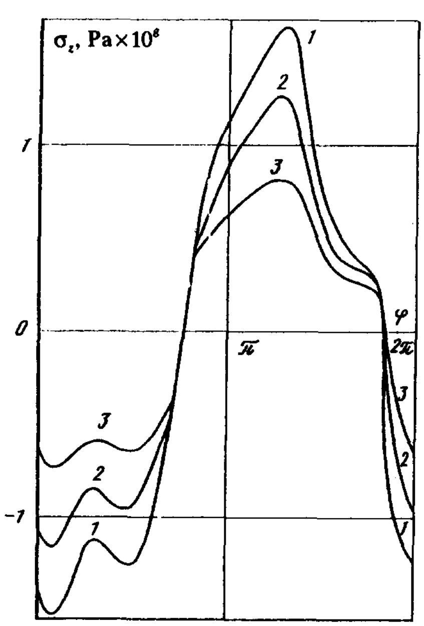

FIG.3.

Figure 1 illustrates the distribution of equivalent stresses found in accordance with the third durability of hypothesis [7] in a cross-section through the middle of the contact zone of the roller with the bar ($\varphi = 0$); Fig. 2 shows the variation of the equivalent stresses along the perimeter of the roller in its surface layers. It is obvious from the graphs that the equivalent stresses become significant only in surface layers of thickness amounting to 0.1–0.15 of the total radius of the roller; in the internal layers, however, these stresses are insignificant. On the surface of the roller, the stresses reach their local maxima in the regions of both the highest and lowest temperatures. Nowhere, however, do the stresses approach the maximum permissible values.

The most intense stresses, in both absolute value and range of variation, are the axial stresses, whose distribution is shown in Fig. 3. At the end of the contact zone of the roller with the bar, the axial stresses on the roller surface reach 1.4×10^8 Pa. Regardless of the frequency of rotation of the roller, the axial stresses change sign in zones corresponding to angular coordinates of 140° and 330°.

The magnitude of the axial stresses and their range of variation depend significantly on the roller rotation frequency: increasing the rate of casting by a factor of six causes the maximum axial stresses to fall to about half their previous value.

Subject to the technological conditions considered here, the thermoelastic stresses do not reach their permissible limits. Hence the non-uniformity of the temperature field cannot damage the rollers.

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